MATH-SHU 350

Recitation 6: Conditional Expectation

Lecturer: Chenlin Gu

Exercise 1. *You throw a dice until you get 6. What is the expected number of throws (including the throw giving 6) conditioned on the event that all throws gave even numbers?*

Exercise 2. *Let $(X_n)_{n\geqslant 1}$ be integrable, i.i.d. random variables, and $S_n = \sum_{i=1}^n X_i$. Find $\mathbb{E}[X_1|X_2]$, $\mathbb{E}[S_n|X_1]$, $\mathbb{E}[S_n|S_{n-1}]$.*

Exercise 3. *Let X, Y be i.i.d. random variables and suppose that $\mathbb{E}[|X|] < \infty$. Show that*

$$\mathbb{E}[X|X+Y] = \mathbb{E}][Y|X+Y] = \frac{X+Y}{2}, \quad \textit{a.s.}\ .$$

Exercise 4. *Recall that*

$$\mathrm{Var}[X|\mathcal{F}] := \mathbb{E}[X^2|\mathcal{F}] - \mathbb{E}[X|\mathcal{F}]^2.$$

Show that

$$\mathrm{Var}[X] = \mathbb{E}[\mathrm{Var}[X|\mathcal{F}]] + \mathrm{Var}[X|\mathcal{F}].$$

Exercise 5. *Suppose X and Y are square-integrable random variables such that $\mathbb{E}[X|Y] = Y$, $\mathbb{E}[Y|X] = X$. Show that $X = Y$ almost surely.*

Exercise 6 (†)**.** *Let (X, Y) be a Gaussian vector in $\mathbb{R}^2$. Find $\mathbb{E}[X|Y]$.*